Section 3.7: Rates of Change in the Natural & Social Sciences

We know that if \( y = f(x) \), then the derivative \( \frac{dy}{dx} \) can be interpreted as the rate of change of \( y \) with respect to \( x \). In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If \( x \) changes from \( x_i \) to \( x_2 \), then the change in \( x \) is

\[
\Delta x = x_2 - x_i
\]

and the corresponding change in \( y \) is

\[
\Delta y = f(x_2) - f(x_i)
\]

The difference quotient

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_i)}{x_2 - x_1}
\]

is the average rate of change of \( y \) with respect to \( x \) over the interval \([x_i, x_2]\) and can be interpreted as the slope of the secant line \( PQ \) in the figure above.

Its limit as \( \Delta x \to 0 \) is the derivative \( f'(x_i) \), which can therefore be interpreted as the instantaneous rate of change of \( y \) with respect to \( x \) or the slope of the tangent line at \( P(x_i, f(x_i)) \).

Using Leibniz notation, we write the process in the form

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

I. Physics

One application to physics we will examine is one-dimensional motion.

If \( s = f(t) \) is the position function of a particle that is moving in a straight line, then \( \Delta s/\Delta t \) represents the average velocity over a time period \( \Delta t \), and \( v = ds/dt \) represents the instantaneous velocity (the rate of change of displacement with respect to time).

The instantaneous rate of change of velocity with respect to time is acceleration:

\[
a(t) = v'(t) = s''(t).
\]

Now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.
Example 1: Suppose the position of an object moving horizontally after $t$ seconds is given by $s = 2t^3 - 2t^2 + 60t$, where $s$ is measure in meters.

A. Find the velocity at time $t$ and after 1 s.

$$\begin{align*}
v(t) &= 6t^2 - 42t + 60 \\
\omega(1) &= 6(1)^2 - 42(1) + 60 \\
&= 24 \text{ m/s}
\end{align*}$$

B. When is the particle at rest?

$$\begin{align*}
v(t) &= 0 \\
6t^2 - 42t + 60 &= 0 \\
\omega(t^2 - 7t + 10) &= 0 \\
\omega(t-2)(t-5) &= 0 \\
t &= 2, 5 \text{ sec}
\end{align*}$$

C. When is the object moving in the positive direction? Negative direction?

$$v > 0 \rightarrow$$

$$v < 0 \leftarrow$$

to find when $v > 0$ or $v < 0$

use the number line and check the sign of $v$ in between the zeros

$$
\begin{array}{c|cc}
& \quad & + & - & + \\
0 & 1 & 2 & 5 \\
\end{array}
$$

$$
\begin{align*}
v(1) &= 24 > 0 \\
v(3) &= -12 < 0 \\
v(6) &= 24 > 0 \\
+ \text{dir}: (0,2) \cup (5,\infty) \\
- \text{dir}: (2,5).
\end{align*}
$$

D. Find the total distance travelled after the first 6 seconds?

Need to account for change in direction.

$$
\begin{align*}
t | 0 & 2 & 5 & 6 \\
s | 0 & 15 & 25 & 36
\end{align*}
$$

$$D = |s(2) - s(0)| + |s(5) - s(2)| + |s(6) - s(5)|$$

$$D = |152 - 0| + |125 - 32| + |136 - 25|$$

$$= 152 + 27 + 11 = 190 \text{ m}$$

E. Draw a diagram that represents the motion of the particle.
F. Find the acceleration after \( t \) seconds and after 1 s:
\[
a(t) = 12t - 42 \text{ m/s}^2
\]
\[
a(1) = 12(1) - 42 = -30 \text{ m/s}^2
\]

G. What is the acceleration when the velocity is 0?
\[
t = 2 \quad t = 5
\]
\[
a(2) = 12 \cdot 2 - 42 = -18 \text{ m/s}^2
\]
\[
a(5) = 12 \cdot 5 - 42 = 18 \text{ m/s}^2
\]

H. Graph the position, velocity, and acceleration functions for \( 0 \leq t \leq 6 \).

I. When is the particle speeding up? Slowing down?

- **Speeding up**: velocity function moves away from zero
  \( v(a) \text{ same sign} \)
- **Slowing down**: velocity function approaches zero
  \( v(a) \text{ are opp in sign} \)

\[
SU (2, 3.5) \quad V(S’<0)
\]
\[
SD (0, 2) \quad V(S’>0)
\]

Example 2: Suppose a stone is thrown vertically upward from a cliff with an initial velocity of 64 ft/s from a height of 96 ft above the ground. The height \( s \) (in feet) of the stone above the ground \( t \) seconds after it is thrown is \( s(t) = -16t^2 + 64t + 96 \).

A. Determine the velocity and acceleration after 1 second?
\[
v(t) = -32t + 64 \text{ ft/sec}
\]
\[
v(1) = -32(1) + 64 = 32 \text{ ft/sec}
\]
\[
a(t) = -32 \text{ ft/sec}^2
\]
\[
a(1) = -32 \text{ ft/sec}^2
\]
B. When does the stone reach the highest point? What is the height of the stone at the highest point?

\[ v(t) = 0 \]
\[ s(t) = -16 \cdot 2^2 + 64 \cdot 2 + 96 \]
\[ = -64 + 128 + 96 \]
\[ = 100 \, \text{ft} \]

The stone reaches a max height of 100 ft after 2 seconds.

C. When does the stone strike the ground? With what speed does the stone strike the ground?

\[ s(t) = 0 \]
\[ -16t^2 + 64t + 96 = 0 \]
\[ -16(t^2 - 4t - 6) = 0 \]
\[ t^2 - 4t - 6 = 0 \]
\[ t = \frac{4 \pm \sqrt{16 - 4(1)(-6)}}{2} \]
\[ = \frac{4 \pm \sqrt{40}}{2} \]
\[ = 5.16 \, \text{sec} \]
\[ v(5.16) = -32(5.16) \]
\[ = -161.28 \, \text{ft/sec} \]

D. When the object travelling fastest? Slowest?

\[ \text{fastest } t \approx 5.16 \, \text{sec} \]
\[ \text{slowest } t = 2 \, \text{sec} \]

II. Biology

Applications of the derivative include analyzing population.

Let \( n = f(t) \) be the number of individuals in an animal or plant population at time \( t \).

The change in the population size between the times \( t = t_1 \) and \( t = t_2 \) is \( \Delta n = f(t_2) - f(t_1) \), and so the average rate of growth during the time period \( t_1 \leq t \leq t_2 \) is

\[
\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]
The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period \( \Delta t \) approach 0:

\[
growth\ rate = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}
\]

Strictly speaking, this is not quite accurate because the actual graph of a population function \( n = f(t) \) would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable.

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as pictured above to the right.

**Example 3:** A certain bacteria population with initially 100 cells doubles every 3 hours.

A. Find a function for the number of bacteria \( n \) after \( t \) hours.

\[
n(t) = N_0 \cdot 2^{t/3} \\
= 100 \cdot 2^{t/3}
\]

B. What was the rate of growth when the bacteria population reached 500 cells?

\[
n(t) = 500 \\
500 = 100 \cdot 2^{t/3} \\
5 = 2^{t/3} \\
\ln 5 = \ln 2^{t/3} \\
\ln 5 = \frac{t}{3} \ln 2 \\
\frac{t}{3} = \ln 5 \\
t = \frac{3 \ln 5}{\ln 2} \\
\approx 6.96 \text{ hr}
\]

The growth rate \( n'(t) \) is given by

\[
n'(t) = 100 \cdot 2^{t/3} \cdot \ln(2) \cdot \frac{d}{dt} \left( \frac{t}{3} \right)
\]

\[
n'(6.94) = 100 \cdot \frac{2^{6.94/3}}{3} \cdot \ln(2) \\
\approx \frac{115 \text{ cells}}{\text{hr}}
\]

When the population reached 500, it grew at a rate of approx 115 cells/hr.
III. Economics

Suppose \( C(x) \) is the total cost that a company incurs in producing \( x \) units of a certain commodity.

The function \( C \) is called a cost function. If the number of items produced is increased from \( x_i \) to \( x_j \), then the additional cost is \( \Delta C = C(x_j) - C(x_i) \), and the average rate of change of the cost is

\[
\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}
\]

The limit of this quantity as \( \Delta x \to 0 \), that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the marginal cost by economists:

\[
\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}
\]

[Since \( x \) often takes on only integer values, it may not make literal sense to let \( \Delta x \) approach 0, but we can replace \( C(x) \) by a smooth approximating function as in the biology analogy]

Taking \( \Delta x = 1 \) and \( n \) large (so that \( \Delta x \) is small compared to \( n \)), we have

\[
C'(n) = C(n + 1) - C(n)
\]

Thus the marginal cost of producing \( n \) units is approximately equal to the cost of producing one more unit [the \( (n + 1) \)st unit].

**Example 4:** A manufacturer produces bolts of a fabric with a fixed width. The cost of producing \( x \) yards of this fabric is \( C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3 \) dollars.

<table>
<thead>
<tr>
<th>A. Find the marginal cost function.</th>
<th>B. Find ( C'(200) ) and explain its meaning.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C'(x) = 12 - .2x + .0015x^2 )/yd</td>
<td>( C'(200) = 12 - .2(200) - .0015(200)^2 )</td>
</tr>
<tr>
<td></td>
<td>( = $32/\text{yd} )</td>
</tr>
<tr>
<td>At 200 yards produced, costs increase by ( $32 ) for an additional yard.</td>
<td></td>
</tr>
</tbody>
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<tr>
<th>C. Compare ( C'(200) ) with the cost of producing the 201st yard.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(201) - C(200) = \left[ 1200 + 12(201) - .1(201)^2 + .0005(201)^3 \right] - \left[ 1200 + 12(200) - .1(200)^2 + .0005(200)^3 \right] )</td>
</tr>
<tr>
<td>( \approx $32.20 ) for the 201st yard.</td>
</tr>
</tbody>
</table>
IV. Further Applications

**Example 5:** The depth \( y \), in feet, of water in Boston harbor is given in terms of \( t \), the number of hours since midnight by

\[
y = 5 + 4.9 \cos\left(\frac{\pi t}{6}\right)
\]

where \( 0 \leq t \leq 24 \).

<table>
<thead>
<tr>
<th>A. Find ( \frac{dy}{dt} ). What does this represent in terms of the water level?</th>
<th>B. How fast is the water level rising (or falling) at 7 am?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dt} = -4.9 \sin\left(\frac{\pi t}{6}\right) )</td>
<td>( y = 5 + 4.9 \cos\left(\frac{\pi}{6}, 7\right) \approx 6.76 \text{ ft} )</td>
</tr>
<tr>
<td>This represents the instantaneous rate of change of depth with respect to time.</td>
<td>( \frac{dy}{dt} = -9.9 \sin\left(\frac{\pi}{6}\right) \approx -1.28 \text{ ft/hr} )</td>
</tr>
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C. When is the water level rising fastest? Falling fastest? Slowest?

Graph \( \frac{dy}{dt} \) over 24 hours

- Rising fastest: 9 am, 3 pm
- Falling fastest: 3 am, 9 pm
- Slowest: 12 am, 6 am, 12 pm, 6 pm

**Example 6:** The volume of a growing spherical cell \( V \), where the radius \( r \) is measured in micrometers \( (1 \mu m = 10^{-6} \text{m}) \)

\[
V = \frac{4}{3} \pi r^3
\]

A. Find the average rate of change from 5 to 8 \( \mu m \), 5 to 6 \( \mu m \), & 5 to 5.1 \( \mu m \).

\[
\begin{align*}
V(5) &= \\
V(5.1) &= \\
V(5) &= \\
V(8) &= \\
\end{align*}
\]